

HARMONIC REFLECTION IN QUASICIRCLES AND WELL-POSEDNESS OF A RIEMANN-HILBERT PROBLEM ON QUASIDISKS

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Dedicated to the memory of Ida Bulat

ABSTRACT. A complex harmonic function of finite Dirichlet energy on a Jordan domain has boundary values in a certain conformally invariant sense, by a construction of H. Osborn. We call the set of such boundary values the Douglas-Osborn space. One may then attempt to solve the Dirichlet problem on the complement for these boundary values. This defines a reflection of harmonic functions. We show that quasicircles are precisely those Jordan curves for which this reflection is defined and bounded.

We then use a limiting Cauchy integral along level curves of Green's function to show that the Plemelj-Sokhotski jump formula holds on quasicircles with boundary data in the Douglas-Osborn space. This enables us to prove the well-posedness of a Riemann-Hilbert problem with boundary data in the Douglas-Osborn space on quasicircles.

1. INTRODUCTION

Consider the following Riemann-Hilbert problem:

Problem 1. *Given a domain in the plane and a function defined on its boundary, find holomorphic functions in the inside and the outside the domain so that the difference of their boundary values is equal to the given function.*

The main issue in this problem is the regularity of the boundary value function (i.e. the boundary data) on one hand, and the regularity of the boundary curve on the other. If the boundary of the domain is a smooth simple closed curve and the boundary function is Lipschitz on the boundary curve, then Problem 1 is the famous Sokhotski-Plemelj jump problem, which was solved independently by Y. Sokhotski [20] and J. Plemelj [12]. In general, if the boundary of the domain is a rough, say fractal-type curve, then given certain functions on the domain, it is not clear how to extend those functions to the boundaries. However, in [19], S. Semmes studied Problem 1 when the boundary curve is a so-called chord-arc curve, and obtained a solution to the problem when the boundary data is in L^2 . Another difficulty that frequently occurs in this context is that Sobolev/Besov spaces are not well-defined on all fractal-type curves, which makes the boundary value problems with data in those spaces problematic. Furthermore, it is difficult to establish the existence of boundary values of

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functions that are defined on either of the sides of such curves, or even to find the appropriate sense in which they are defined. Nevertheless, in problems connected to quasiconformal Teichmüller theory as well as those arising in connection to the mathematical foundations of conformal field theory, one inevitably confronts Problem 1 where the domain is a quasidisk. Indeed our particular choice of the Riemann-Hilbert problem here is strongly motivated by applications in the aforementioned fields [14].

A quasidisk is the image of the unit disk under a quasiconformal map of the plane. The boundaries of quasidisks, called quasicircles, can be very rough curves with Hausdorff dimensions arbitrarily close to 2; examples include fractal-type curves such as snowflakes. The authors (together with D. Radnell) had previously studied Problem 1 in the case when the boundary curve is assumed to be a Weil-Petersson class quasicircle [15], and Schippers and Staubach extended the study of Problem 1 to d -regular quasicircles (which are a generalization of Ahlfors-regular quasicircles) where the boundary function was assumed to belong to a certain Besov space [18].

Here we consider Problem 1 when the domain is a general quasidisk and remove the assumption of d -regularity which was imposed in [18]. Moreover, as opposed to [18], we obtain the solution for an optimal class of boundary data and show that our solution yields yet another characterization of quasidisks. More precisely, to bypass the regularity issues of the boundary, we use a particular reflection of harmonic functions, and a limiting Cauchy integral. The reflection of harmonic functions exists and is bounded if and only if the domain is a quasidisk. We then use this to demonstrate that Problem 1 has a unique solution for a space of functions (which we call the Douglas-Osborn space) defined on the boundary of a quasidisk, and that the solution depends continuously on the boundary data. Hence the Riemann-Hilbert problem above is well-posed.

In this connection we would like to mention that the solution of this boundary value problem could be compared with Poincaré’s lemma. That is, the existence of solutions to $df = \alpha$ for a closed differential form α (exactness) is intimately connected to the topology of the underlying domain (or manifold). Here, the existence of a solution of the Riemann-Hilbert problem is intimately connected, rather, to the regularity (e.g. the Hausdorff dimension) of its boundary.

Here is an explanation of our approach. Recall the classical Plemelj-Sokhotski jump formula [12], which asserts that for a complex function h (say, continuous) on a smooth closed Jordan curve Γ bounding two regions Ω^\pm in the plane (where we assume Ω^- is unbounded), if

$$(1.1) \quad h_\pm(z) = \pm \frac{1}{2\pi i} \int_\Gamma \frac{h(\zeta)}{\zeta - z} d\zeta \quad z \in \Omega^\pm$$

then

$$h(z) = \lim_{\Omega^+ \ni w \rightarrow z} h_+(w) + \lim_{\Omega^- \ni w \rightarrow z} h_-(w).$$

For smooth boundaries and regular boundary values, this provides a solution to Problem 1, as mentioned above; the term “jump” refers to the discontinuity in the boundary values of

the Cauchy integral. Henceforth, we will use the equivalent formulation of the jump problem which requires an expression for h as a sum rather than a difference. To solve Problem 1 for quasicircles, we prove a jump formula for boundary data that are the limiting values of complex harmonic functions of bounded Dirichlet energy. The choices are naturally dictated and in some sense optimal. We also give a characterization of quasidisks in terms of the existence of a bounded reflection on harmonic functions.

The boundary values are defined using the following conformally invariant construction of H. Osborn [11]. Let Γ be any Jordan curve, with Green's function g_p with singularity at p . Let $C_{p,\epsilon}$ denote the level curves $g_p(z) = -\epsilon$, and for $q \in \Gamma$ let $\gamma_{p,q}$ denote the unique orthogonal curve beginning at p and terminating at q . Osborn showed that for any Jordan curve, a harmonic function of finite Dirichlet energy in a connected component of the complement has boundary values a.e. on Γ with respect to harmonic measure, where these boundary values are approached along $\gamma_{p,q}$.

In this paper, we show that if Γ is a quasicircle bounding two domains Ω^\pm in the sphere, the space of functions obtained as boundary values of complex harmonic functions of finite Dirichlet energy on Ω^+ or Ω^- are the same. We call this space the Douglas-Osborn space. Thus we may define a “harmonic reflection” in quasicircles as follows. Given h_{Ω^+} in the Dirichlet space of Ω^+ , restrict to Γ in the sense of Osborn, and then let h_{Ω^-} be the unique function in the Dirichlet space of Ω^- with boundary values h . This is well-defined for quasicircles by the above, and furthermore is bounded. The same holds for the inverse reflection.

We also define a boundary Cauchy integral for h in the Douglas-Osborn space of a quasicircle Γ , by performing the Cauchy integral (1.1) of the harmonic extension of h to Ω^+ along the level curves $C_{p,\epsilon}$ and letting ϵ go to zero. We show that one obtains the same result if one extends instead to Ω^- and uses level curves in Ω^- . Finally, we show that the outcome of this limiting Cauchy integral satisfies the Plemelj-Sokhotski jump formula.

We conclude the introduction with a remark on the results of Osborn. Osborn's work was concerned with extending the ideas of J. Douglas used in his resolution of the Plateau problem and making them rigorous. In particular, Douglas expressed the Dirichlet norm of a harmonic function in terms of an integral over the boundary of a domain, and in some sense his solution was closely related to the Dirichlet principle. For a historical discussion see J. Gray and M. Micallef [7]. Osborn's beautiful paper extended this to finitely-connected domains bounded by Jordan curves, among other striking results. It is with this historical development in mind that we named the set of boundary values after Douglas and Osborn.

2. REFLECTION OF HARMONIC FUNCTIONS

2.1. Notation and terminology. In this paper, \mathbb{C} is the complex plane, $\overline{\mathbb{C}}$ is the Riemann sphere, $\mathbb{D}^+ = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{D}^- = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. We will denote the boundary of a domain Ω by $\partial\Omega$, and $\partial\mathbb{D}^\pm$ by \mathbb{S}^1 . Closure will be denoted by cl. Furthermore, generic constants appearing in all the estimates will be denoted by C even though they may be different in the same line or from line to line.

Throughout the paper, we will deal with Jordan curves Γ in $\overline{\mathbb{C}}$. We always assume that Γ is given an orientation, and denote the components of the complement of Γ in $\overline{\mathbb{C}}$ by Ω^+ and Ω^- . We choose Ω^+ to be the component such that Γ is positively oriented with respect to Ω^+ . In the case that Γ is bounded, we always assume that Ω^+ is the bounded component of the complement, so that the orientation of Γ is uniquely determined in this case. In Section 2, the orientation of the curve is only necessary to fix a meaning of the notation Ω^\pm . In Section 3 the orientation affects the signs of the integrals. Finally, the phrase “ Γ bounds Ω ” does not imply that Ω is a bounded domain.

We shall also consider harmonic and holomorphic functions on domains containing ∞ . To say that h is harmonic or holomorphic on a domain Ω containing ∞ is to say that h is harmonic or holomorphic on $\Omega \setminus \{\infty\}$ and $h(1/z)$ is harmonic or holomorphic on a neighbourhood of 0. In this paper, a conformal map between open connected subsets D and Ω of $\overline{\mathbb{C}}$ is taken to mean a map which is a meromorphic bijection between D and Ω . (In particular, it can have at worst a simple pole, possibly at ∞).

2.2. Osborn space of boundary values. Throughout the paper, we choose norms which are conformally invariant, so that the behaviour at ∞ does not require special treatment.

Let dA denote the Lebesgue measure, and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$, $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$ denote the Wirtinger operators. Define the complex Dirichlet space by

$$\mathcal{D}_{\text{harm}}(\Omega) = \left\{ h \text{ is harmonic} : \iint_{\Omega} (|\partial h|^2 + |\bar{\partial} h|^2) dA < \infty \right\}$$

with semi-norm

$$(2.1) \quad \|h\|_{\mathcal{D}_{\text{harm}}(\Omega)}^2 = \iint_{\Omega} (|\partial h|^2 + |\bar{\partial} h|^2) dA.$$

This is of course not a norm since if two functions h_1 and h_2 differ by a constant, then $\|h_1\|_{\mathcal{D}_{\text{harm}}(\Omega)} = \|h_2\|_{\mathcal{D}_{\text{harm}}(\Omega)}$. Note that this space is conformally invariant, so that if $g : D \rightarrow \Omega$ is any biholomorphism then

$$\|h \circ g\|_{\mathcal{D}_{\text{harm}}(D)} = \|h\|_{\mathcal{D}_{\text{harm}}(\Omega)}.$$

For $p \in \Omega$, we will also consider the pointed Dirichlet space $\mathcal{D}_{\text{harm},p}(\Omega)$, which is the same set of functions but with the actual norm

$$\|h\|_{\mathcal{D}_{\text{harm},p}(\Omega)}^2 = |h(p)|^2 + \iint_{\Omega} (|\partial h|^2 + |\bar{\partial} h|^2) dA.$$

This also satisfies a conformal invariance: if $g : D \rightarrow \Omega$ is any biholomorphism then

$$\|h \circ g\|_{\mathcal{D}_{\text{harm},p}(D)} = \|h\|_{\mathcal{D}_{\text{harm},g(p)}(\Omega)}.$$

Now let Ω be a Jordan domain as above with boundary Γ . We define a set of boundary values of $\mathcal{D}_{\text{harm}}(\Omega)$, following Osborn [11], as follows. First some notation is required. Fix a point $p \in \Omega$ and let g_p be Green's function on Ω with singularity at p . For $\epsilon > 0$ let $C_{p,\epsilon}$ be the level curve $g_p(z) = -\epsilon$. For $q \in \Gamma$, let $\gamma_{p,q}$ denote the unique ray which is orthogonal to the set of curves $C_{p,\epsilon}$, starting at p and terminating at q .

Another characterization of the curves $\gamma_{p,q}$ and $C_{p,\epsilon}$ will be useful. Let $f : \mathbb{D}^+ \rightarrow \Omega$ be a conformal map such that $f(0) = p$. By the Osgood-Carathéodory theorem, f extends to a homeomorphism of \mathbb{S}^1 onto Γ . In that case, each curve $\gamma_{p,q}$ is the image of the ray $t \mapsto tf^{-1}(q)$ for $t \in [0, 1]$. Also $C_{p,\epsilon}$ is the image of $|z| = r$ under f for $r = e^{-\epsilon}$. This follows

directly from the conformal invariance of Green's function. One can also characterize the curves $\gamma_{p,q}$ as the hyperbolic geodesic ray with end-points p and q .

In [11], Osborn showed that for any $h \in \mathcal{D}_{\text{harm}}(\Omega)$, the limit

$$\lim_{z \rightarrow q} h(z)$$

exists almost everywhere. We provide the proof in the simply connected case, as a consequence of classical results.

The Sobolev space $H^{\frac{1}{2}}(\mathbb{S}^1)$ is the space of functions $h \in L^2(\mathbb{S}^1)$ such that $\sum_{n=-\infty}^{\infty} (1 + |n|^2)^{\frac{1}{2}} |\hat{h}(n)|^2 < \infty$, where $\hat{h}(n) := \frac{1}{2\pi} \int_{\mathbb{S}^1} h(t) e^{-int} dt$. The norm of h in $H^{\frac{1}{2}}(\mathbb{S}^1)$ is given by

$$\|h\|_{H^{\frac{1}{2}}(\mathbb{S}^1)}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^2)^{\frac{1}{2}} |\hat{h}(n)|^2.$$

Here we consider a slight modification of this space, denoted by $\mathcal{H}(\mathbb{S}^1)$, which consists of L^2 functions on the circle for which

$$(2.2) \quad \|h\|_{\mathcal{H}(\mathbb{S}^1)}^2 = \sum_{n=-\infty}^{\infty} |n| |\hat{h}(n)|^2 < \infty.$$

This is of course a semi-norm. $\mathcal{H}(\mathbb{S}^1)$ and $H^{\frac{1}{2}}(\mathbb{S}^1)$ are the same set of functions [17]. We also consider the norm

$$(2.3) \quad \|h\|_{\mathcal{H}_0(\mathbb{S}^1)}^2 = |\hat{h}(0)|^2 + \sum_{n=-\infty}^{\infty} |n| |\hat{h}(n)|^2 < \infty.$$

One can easily show that the norms of $H^{\frac{1}{2}}(\mathbb{S}^1)$ and $\mathcal{H}_0(\mathbb{S}^1)$ are equivalent, see e.g. [17].

In the following discussion, and the rest of this paper, the term “capacity” will always refer to logarithmic capacity, and similarly for “outer capacity”. The space $\mathcal{H}(\mathbb{S}^1)$ consists precisely of the boundary values of functions in the Dirichlet space $\mathcal{D}_{\text{harm},0}(\mathbb{D}^+)$, and the norms are equal [10]. We briefly outline this correspondence here. The Fourier series of any element $h(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{h}(n) e^{in\theta} \in \mathcal{H}(\mathbb{S}^1)$ converges everywhere on \mathbb{S}^1 except possibly on a set of outer capacity zero, see A. Zygmund, [21, Theorem 11.3, Chapter XIII]. Therefore the series

$$(2.4) \quad H(z) = \sum_{n=0}^{\infty} \hat{h}(n) z^n + \sum_{n=1}^{\infty} \hat{h}(-n) \bar{z}^n$$

converges to a sum of a holomorphic and an anti-holomorphic function and is thus complex harmonic. Furthermore, a standard computation shows that

$$\|h\|_{\mathcal{H}(\mathbb{S}^1)} = \|H\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+)} \quad \text{and} \quad \|h\|_{\mathcal{H}_0(\mathbb{S}^1)} = \|H\|_{\mathcal{D}_{\text{harm},0}(\mathbb{D}^+)}.$$

By Abel's theorem and the convergence of h in the sense above, $\lim_{t \rightarrow 1^-} H(te^{i\theta}) = h(e^{i\theta})$ for all $e^{i\theta} \in \mathbb{S}^1$ except on a set of zero outer capacity; that is, the boundary values agree with h in the sense of Osborn except on this set. Conversely, an element $H(z) \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ has the form (2.4) with coefficients $\hat{h}(n)$ satisfying (2.2). One thus obtains an element h of $\mathcal{H}(\mathbb{S}^1)$ by replacing z with $e^{i\theta}$, and the argument above can be repeated to show that h is the limit of H on the boundary in the same sense.

Summarizing, we have the following result.

Theorem 2.1. *Every $H \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ has boundary values (in the sense of Osborn) except on a subset of \mathbb{S}^1 with outer capacity zero, and the restriction to the boundary belongs to $\mathcal{H}(\mathbb{S}^1)$. Conversely, every $h \in \mathcal{H}(\mathbb{S}^1)$ has a unique complex harmonic extension $H \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ whose boundary values coincide with h in the same sense. The restriction and extension are isometries with respect to $\|\cdot\|_{\mathcal{H}_0(\mathbb{S}^1)}$ and $\|\cdot\|_{\mathcal{D}_{\text{harm},0}(\mathbb{D}^+)}$. The restriction and extension are also bounded with respect to the semi-norms $\|\cdot\|_{\mathcal{H}(\mathbb{S}^1)}$ and $\|\cdot\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+)}$.*

Remark 2.2. Any set $E \subset \mathbb{S}^1$ of outer capacity zero is contained in a Borel set $I \subset \mathbb{S}^1$ of capacity zero. To see this, for any n let V_n be an open subset of \mathbb{C} containing E such that the capacity of V_n is less than $1/n$. If we let $U_n = V_n \cap \mathbb{S}^1$, then U_n is an open set in \mathbb{S}^1 (and a Borel set in \mathbb{C}). Since $U_n \subset V_n$ the capacity of U_n is less than $1/n$ by [16, 5.1.2(a)]. Now if we set $I = \bigcap_{n=1}^{\infty} U_n$ (which is obviously a Borel set), then since $I \subseteq U_n$ for all n (again using [16, 5.1.2(a)]) we have that the capacity of I is less than or equal to $1/n$ for all n . Hence I is the desired set of capacity zero which contains E by construction.

Note that it is possible to take I to be a G_δ set. To see this, in the above construction one could choose $U_n = V_n \cap \{z : 1 - 1/n < |z| < 1 + 1/n\}$.

The extension and restriction map is described explicitly as follows. Given $h \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$, it has a decomposition $h = u + \bar{v}$ for holomorphic functions

$$u(z) = \sum_{n=0}^{\infty} u_n z^n, \quad v(z) = \sum_{n=1}^{\infty} v_n z^n.$$

In that case $h(e^{i\theta}) = \sum_{n=0}^{\infty} u_n e^{in\theta} + \sum_{n=1}^{\infty} \bar{v}_n e^{-in\theta}$ and the extension is obtained similarly.

Remark 2.3. We also observe that Theorem 2.1 applies equally to $\mathcal{D}_{\text{harm}}(\mathbb{D}^-)$. Note that the reflection

$$\begin{aligned} \mathfrak{R}(\mathbb{S}^1) : \mathcal{D}_{\text{harm}}(\mathbb{D}^+) &\rightarrow \mathcal{D}_{\text{harm}}(\mathbb{D}^-) \\ H(z) &\mapsto H(1/\bar{z}) \end{aligned}$$

is a semi-norm preserving isomorphism between $\mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ and $\mathcal{D}_{\text{harm}}(\mathbb{D}^-)$. In fact it is an isometry between $\mathcal{D}_{\text{harm},0}(\mathbb{D}^+)$ and $\mathcal{D}_{\text{harm},\infty}(\mathbb{D}^-)$.

We will also denote the inverse of $\mathfrak{R}(\mathbb{S}^1)$ with the same notation.

Thus given a Jordan domain Ω and fixed $p \in \Omega$ we obtain boundary values for $H \in \mathcal{D}_{\text{harm}}(\Omega)$ as follows.

Theorem 2.4 (Osborn [11]). *Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ and let Ω be either of the components of the complement. Fix $p \in \Omega$. For any $H \in \mathcal{D}_{\text{harm}}(\Omega)$,*

$$h(q) = \lim_{z \rightarrow q} H(z), \quad z \in \gamma_{p,q}$$

exists for almost all q with respect to the harmonic measure on Γ with respect to p and Ω .

Proof. Let $f : \mathbb{D}^+ \rightarrow \Omega$ be a conformal map such that $f(0) = p$. Since f extends homeomorphically to a map $F : \text{cl } \mathbb{D}^+ \rightarrow \text{cl } \Omega$, we have that the limits

$$\lim_{\gamma_{p,q} \ni z \rightarrow q} H(z) = \lim_{f^{-1}(\gamma_{p,q}) \ni w \rightarrow f^{-1}(q)} H \circ F(w)$$

exist except on a set of outer capacity zero on \mathbb{S}^1 by Theorem 2.1, and thus also exist except on a set of capacity zero by Remark 2.2. A set of capacity zero on \mathbb{S}^1 has zero harmonic measure with respect to \mathbb{D}^+ (see T. Ransford [16, Theorem 4.3.6]). Since the harmonic measure of a set $E \subset \partial\Omega$ with respect to Ω is the harmonic measure of $F^{-1}(E)$ with respect to \mathbb{S}^1 (see J. Garnett and D. Marshall [6, I.3]), this completes the proof. \square

Remark 2.5. Note that the proof shows that the set of points on the boundary for which the radial limits exist is preserved by a conformal map.

The conformal invariance of $\mathcal{D}_{\text{harm}}(\Omega)$ yields in particular its invariance under biholomorphisms $T : \Omega \rightarrow \Omega$. Furthermore, since sets of harmonic measure 0 are also conformally invariant (that is, independent of the point p), it immediately follows that the possible set of functions on the boundary (defined up to sets of harmonic measure 0) is independent of the choice of $p \in \Omega$. Thus we can make the following definition.

Definition 2.6. For a Jordan curve Γ bounding a domain Ω , let $\mathcal{H}_{\pm}(\Gamma)$ denote the set of functions obtained as boundary values of functions in $\mathcal{D}_{\text{harm}}(\Omega^{\pm})$.

It is important to observe that $\mathcal{H}_{\pm}(\Gamma)$ are quite large classes of functions. For example, $\mathcal{H}_{+}(\Gamma)$ contains $H \circ f^{-1}$ for all H which are C^1 on the closed disk $\text{cl}\mathbb{D}^+$ where $f : \mathbb{D}^+ \rightarrow \Omega^+$ is a conformal bijection.

Note that harmonic measure on the inside and outside might be incomparable, and thus we must carry the distinction between boundary values obtained from the inside and outside. On the other hand, by applying the reflection $\mathfrak{R}(\mathbb{S}^1)$, which preserves harmonic measure with respect to 0 and ∞ , we have that

$$\mathcal{H}_{\pm}(\mathbb{S}^1) = \mathcal{H}(\mathbb{S}^1).$$

Remark 2.7. The existence of boundary values can also be viewed in the following way. The limit exists on the ideal boundary of Ω . So long as we consider only domains bounded by Jordan curves, then by the Osgood-Carathéodory theorem, there is a one-to-one correspondence between points on the ideal boundary and the boundary $\partial\Omega$ in \mathbb{C} .

Theorem 2.1 has the following immediate consequence:

Theorem 2.8. Let Γ be a Jordan curve in $\overline{\mathbb{C}}$ bounding domains Ω^{\pm} . Any $h \in \mathcal{H}_{\pm}(\Gamma)$ is the boundary values of a unique $H \in \mathcal{D}_{\text{harm}}(\Omega^{\pm})$.

Finally, we conclude with a result which will be necessary in the next section. It is a consequence of a theorem due to N. Arcozzi and R. Rochberg [1], which states that if $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a quasisymmetry and I is a closed subset of \mathbb{S}^1 , then there is a constant $C > 0$ depending only on ϕ such that $\frac{1}{C} \text{cap}(I) \leq \text{cap}(\phi(I)) \leq C \text{cap}(I)$. In [1] this result is stated to be a corollary of a theorem due to A. Beurling and L. Ahlfors [2], and an independent proof is also given.

Theorem 2.9. If $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a quasisymmetry then ϕ takes Borel sets of capacity zero to Borel sets of capacity zero.

Proof. First we observe that, since ϕ and ϕ^{-1} are homeomorphisms, they take Borel sets to Borel sets. Denoting the capacity of a set A by $\text{cap}(A)$, let E be a Borel subset of \mathbb{S}^1 with $\text{cap}(E) = 0$.

Assume, by way of contradiction, that $\text{cap}(\phi(E)) > 0$. Then since the capacity of $\phi(E)$ is the

supremum of the capacities of all compact subsets K of $\phi(E)$ (see [16, Theorem 5.1.2(b)]), there is a compact $K \subseteq \phi(E)$ such that $\text{cap}(K) > 0$. Moreover, $\phi^{-1}(K)$ is closed (in fact compact) in \mathbb{S}^1 , since ϕ^{-1} is also a quasimetry. Therefore the result of Arcozzi and Rochberg mentioned above yields that there is a constant $C > 0$ depending only on ϕ^{-1} such that $\text{cap}(\phi^{-1}(K)) \geq \frac{1}{C} \text{cap}(K) > 0$. But since $\phi^{-1}(K) \subseteq E$, Theorem 5.1.2(b) in [16] would then imply that $\text{cap}(E) > 0$, which is a contradiction. Thus $\text{cap}(\phi(E)) = 0$. \square

2.3. Harmonic reflection in quasicircles. We require the following results characterizing quasimetrics. For $h \in \mathcal{H}_\pm(\Gamma)$ and a homeomorphism $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, define

$$C_\phi h = h \circ \phi.$$

If we furthermore assume that $h \circ \phi \in L^1(\mathbb{S}^1)$ then we can define

$$\hat{C}_\phi h = h \circ \phi - \frac{1}{2\pi} \int_{\mathbb{S}^1} h \circ \phi(e^{i\theta}) d\theta.$$

Following S. Nag and D. Sullivan, we define

$$\dot{\mathcal{H}}(\mathbb{S}^1) = \left\{ h \in \mathcal{H}(\mathbb{S}^1) : \hat{h}(0) = 0 \right\}.$$

We observe that the restriction of the seminorm $\|\cdot\|_{\mathcal{H}(\mathbb{S}^1)}$ to $\dot{\mathcal{H}}(\mathbb{S}^1)$ is a norm.

Moreover, note that the restriction and extension are isometries with respect to $\|\cdot\|_{\mathcal{H}(\mathbb{S}^1)}$ and $\|\cdot\|_{\mathcal{D}_{\text{harm}}(\mathbb{D}^+)}$, if we assume that the extension is zero at 0. We then have the following theorem of Nag-Sullivan [10].

Theorem 2.10 ([10], Theorem 3.1). *Let $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. Then ϕ is a quasimetry if and only if $h \circ \phi \in \mathcal{H}(\mathbb{S}^1)$ for all $h \in \mathcal{H}(\mathbb{S}^1)$ and $\hat{C}_\phi : \dot{\mathcal{H}}(\mathbb{S}^1) \rightarrow \dot{\mathcal{H}}(\mathbb{S}^1)$ is bounded with respect to the norm $\|\cdot\|_{\mathcal{H}(\mathbb{S}^1)}$.*

Remark 2.11. The statement of the Theorem 3.1 in [10] omitted the boundedness condition in one direction, which was used in their proof. We have thus re-worded the statement of their theorem slightly.

Remark 2.12. An equivalent way to state the theorem is that a homeomorphism ϕ is a quasimetry if and only if $h \circ \phi \in \mathcal{H}(\mathbb{S}^1)$ for all $h \in \mathcal{H}(\mathbb{S}^1)$ and C_ϕ is bounded with respect to the semi-norm $\|\cdot\|_{\mathcal{H}(\mathbb{S}^1)}$.

Theorem 2.13. *Let $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a homeomorphism. Then ϕ is a quasimetry if and only if C_ϕ takes $\mathcal{H}(\mathbb{S}^1)$ into $\mathcal{H}(\mathbb{S}^1)$ and is bounded with respect to $\|\cdot\|_{\mathcal{H}_0(\mathbb{S}^1)}$.*

Proof. The fact that ϕ a quasimetry implies that C_ϕ is a bounded map into $\mathcal{H}(\mathbb{S}^1)$ was proven by the authors in [17, Theorem 2.7].

To prove the converse, assume that C_ϕ is bounded from $\mathcal{H}(\mathbb{S}^1)$ into $\mathcal{H}(\mathbb{S}^1)$ with respect to $\|\cdot\|_{\mathcal{H}_0(\mathbb{S}^1)}$. Since $\mathcal{H}(\mathbb{S}^1) \subset L^2(\mathbb{S}^1) \subset L^1(\mathbb{S}^1)$ the average

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} h \circ \phi(e^{i\theta}) d\theta$$

is defined for any $h \in \mathcal{H}(\mathbb{S}^1)$. Thus for any $h \in \dot{\mathcal{H}}(\mathbb{S}^1)$ we can meaningfully make the following estimate:

$$\begin{aligned} \|\hat{C}_\phi h\|_{\mathcal{H}(\mathbb{S}^1)} &= \|C_\phi h\|_{\mathcal{H}(\mathbb{S}^1)} \leq \|C_\phi h\|_{\mathcal{H}_0(\mathbb{S}^1)} \\ &\leq M \|h\|_{\mathcal{H}_0(\mathbb{S}^1)} = M \|h\|_{\mathcal{H}(\mathbb{S}^1)}. \end{aligned}$$

Thus the claim follows from Theorem 2.10. \square

Theorem 2.13 is a strengthening of Theorem 2.10 in one direction and a weakening in the other.

This leads to the following interesting characterization of quasicircles. Let $\mathcal{D}_{\text{harm}}(\Omega^+)_r$ temporarily denote the set of $h \in \mathcal{D}_{\text{harm}}(\Omega^+)$ whose boundary values are equal to an element of $\mathcal{H}_-(\Gamma)$ almost everywhere with respect to harmonic measure induced by Ω^- . We have a well-defined “reflection” $\mathfrak{R}^{+-}(\Gamma) : \mathcal{D}_{\text{harm}}(\Omega^+)_r \rightarrow \mathcal{D}_{\text{harm}}(\Omega^-)$ given by letting $\mathfrak{R}^{+-}(\Gamma)h$ be the unique harmonic function in $\mathcal{D}_{\text{harm}}(\Omega^-)$ whose boundary values equal those of h almost everywhere. We can also define a reflection $\mathfrak{R}^{-+}(\Gamma)$ similarly.

Theorem 2.14. *Let Γ be a Jordan curve and let Ω^\pm denote the complements of Γ in $\overline{\mathbb{C}}$. The following are equivalent:*

- (1) *for all $h \in \mathcal{D}_{\text{harm}}(\Omega^+)$, the boundary values of h are in $\mathcal{H}_-(\Gamma)$, and $\mathfrak{R}^{+-}(\Gamma) : \mathcal{D}_{\text{harm}}(\Omega^+) \rightarrow \mathcal{D}_{\text{harm}}(\Omega^-)$ is a bounded operator;*
- (2) *for all $h \in \mathcal{D}_{\text{harm}}(\Omega^-)$, the boundary values of h are in $\mathcal{H}_+(\Gamma)$, and $\mathfrak{R}^{-+}(\Gamma) : \mathcal{D}_{\text{harm}}(\Omega^-) \rightarrow \mathcal{D}_{\text{harm}}(\Omega^+)$ is a bounded operator;*
- (3) *Γ is a quasicircle.*

Finally, if Γ is a quasicircle, then for any $h \in \mathcal{D}_{\text{harm}}(\Omega^+)$, the boundary values of $\mathfrak{R}^{+-}(\Gamma)h$ exist and agree with those of h , except on a set K , which is simultaneously the image of a subset of \mathbb{S}^1 of capacity zero under a conformal map $f : \mathbb{D}^+ \rightarrow \Omega^+$, and the image of a subset of \mathbb{S}^1 of capacity zero under a conformal map $g : \mathbb{D}^- \rightarrow \Omega^-$. In particular, K is a set of harmonic measure zero with respect to both Ω^+ and Ω^- .

The same claim holds for $h \in \mathcal{D}_{\text{harm}}(\Omega^-)$ and $\mathfrak{R}^{-+}(\Gamma)h$.

Proof. First, observe that all three conditions hold for $\Gamma = \mathbb{S}^1$ by Theorem 2.1 and Remark 2.3. We will use this repeatedly in what follows.

Assume that Γ is a quasicircle. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ and $g : \mathbb{D}^- \rightarrow \Omega^-$ be conformal maps, which must have quasiconformal extensions to $\overline{\mathbb{C}}$. Setting $\phi = g^{-1} \circ f$, we have that ϕ^{-1} is a quasisymmetry. Now let $h \in \mathcal{H}_+(\Gamma)$ and let H_+ be its harmonic extension in $\mathcal{D}_{\text{harm}}(\Omega^+)$. We will show that h has an extension $H_- \in \mathcal{D}_{\text{harm}}(\Omega^-)$ and that the map $H_+ \mapsto H_-$ is bounded from $\mathcal{D}_{\text{harm}}(\Omega^+)$ to $\mathcal{D}_{\text{harm}}(\Omega^-)$.

Denote the extension from $\mathcal{H}(\mathbb{S}^1)$ to $\mathcal{D}_{\text{harm}}(\mathbb{D}^\pm)$ by e_\pm and the map from $\mathcal{D}_{\text{harm}}(\mathbb{D}^\pm)$ to the boundary values in $\mathcal{H}(\mathbb{S}^1)$ by r_\pm . We have that r_\pm and e_\pm are bounded maps with respect to the $\mathcal{H}(\mathbb{S}^1)$ norm. Denote also

$$\begin{aligned} C_f : \mathcal{D}_{\text{harm}}(\Omega^+) &\rightarrow \mathcal{D}_{\text{harm}}(\mathbb{D}^+) \\ h &\mapsto h \circ f \end{aligned}$$

and

$$\begin{aligned} C_{g^{-1}} : \mathcal{D}_{\text{harm}}(\mathbb{D}^-) &\rightarrow \mathcal{D}_{\text{harm}}(\Omega^-) \\ h &\mapsto h \circ g^{-1} \end{aligned}$$

which preserve the semi-norm by change of variables. By Theorem 2.13, we then have that $C_{g^{-1}}e_-C_{\phi^{-1}}r_+C_fH_+$ is in $\mathcal{D}_{\text{harm}}(\Omega^-)$ and the map $C_{g^{-1}}e_-C_{\phi^{-1}}r_+C_f$ is bounded.

Now set $H_- = C_{g^{-1}}e_-C_{\phi^{-1}}r_+C_fH_+$; it remains to be shown that the boundary values of H_- and h agree except on a set K with the specified properties.

First observe that the boundary values $\widetilde{H_+ \circ f}$ of $H_+ \circ f \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$ exist in $\mathcal{H}(\mathbb{S}^1)$ in the sense of Osborn, except on a Borel set E of capacity zero, by Theorem 2.1 and Remark 2.2. By Theorem 2.9, $\phi(E)$ also has capacity zero, since ϕ is a quasismymetry. Now consider $\widetilde{H_+ \circ f \circ \phi^{-1}}$. By Theorem 2.13, this is in $\mathcal{H}(\mathbb{S}^1)$, and by Theorem 2.1 it has an extension to a function $G \in \mathcal{D}_{\text{harm}}(\mathbb{D}^-)$ (say) whose boundary values agree with $\widetilde{H_+ \circ f \circ \phi^{-1}}$ except on a Borel set F of capacity zero in \mathbb{S}^1 . If we set $I = F \cup \phi(E)$, then we have that I has zero capacity. To see this, we observe that since both F and $\phi(E)$ have capacity zero and are both bounded Borel sets, then their outer capacity which is by Choquet's theorem [3] equal to their capacity, is also equal to zero. Therefore using the subadditivity of the outer capacity under countable unions [5, Theorem 2.1.9] and Choquet's theorem, we have that $\text{cap}(I) = 0$.

Now if we set $K = g(I)$, then K is the image of a set of capacity zero under g , and the function $G \circ g^{-1} = H_-$ is in $\mathcal{D}_{\text{harm}}(\Omega^-)$. Furthermore the boundary values agree with $\widetilde{H_+ \circ f \circ \phi^{-1} \circ g^{-1}} = H_+$ on $\Gamma \setminus g(I) = \Gamma \setminus K$ (see Remark 2.5). On the other hand, $K = g(I) = g(F) \cup g(\phi(E)) = f(\phi^{-1}(F)) \cup f(E) = f(\phi^{-1}(F) \cup E)$. Since ϕ^{-1} is also a quasismymetry, by the argument applied above $I' = \phi^{-1}(F) \cup E$ has capacity zero and hence $K = f(I')$ is the image of a subset of \mathbb{S}^1 of capacity zero under f as claimed.

As we mentioned in the proof of Theorem 2.4, subsets of \mathbb{S}^1 of logarithmic capacity zero are null-sets with respect to the harmonic measure. Therefore [6, I.3, equation (3.3)] yields that $K = g(I)$ has harmonic measure zero with respect to Ω^- and since $K = f(I')$, it also has harmonic measure zero with respect to Ω^+ . Thus we have shown that (3) implies (1), and also that (3) implies the final claim. The proof that (3) implies (2) and the final claim is similar, hence omitted. Thus, once the equivalence of (1), (2), and (3) is demonstrated, the proof of the theorem will be complete.

We show that (1) implies (3); the proof that (2) implies (3) is similar. Assuming (1), we define the bounded reflection $\mathfrak{R}^{+-}(\Gamma) : \mathcal{D}_{\text{harm}}(\Omega^+) \rightarrow \mathcal{D}_{\text{harm}}(\Omega^-)$. Let f and g be conformal maps of \mathbb{D}^+ and \mathbb{D}^- onto Ω^+ and Ω^- respectively. By the Osgood-Carathéodory theorem, f and g extend to homeomorphisms of \mathbb{S}^1 to Γ . Thus we may define a homeomorphism $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ by $\phi = g^{-1} \circ f$. For any $H_+ \in \mathcal{D}_{\text{harm}}(\mathbb{D}^+)$, $C_g \mathfrak{R}^{+-}(\Gamma) C_{f^{-1}} H_+$ is in $\mathcal{D}_{\text{harm}}(\mathbb{D}^-)$. Furthermore, for any $h \in \mathcal{H}(\mathbb{S}^1)$, by comparing boundary values as above, $C_{\phi^{-1}} h = r_- C_g \mathfrak{R}^{+-}(\Gamma) C_{f^{-1}} e_+ h$. Thus

$$C_{\phi^{-1}} = r_- C_g \mathfrak{R}^{+-}(\Gamma) C_{f^{-1}} e_+$$

and since all maps on the right hand side are bounded, we conclude that $C_{\phi^{-1}}$ is a bounded operator on $\mathcal{H}(\mathbb{S}^1)$ with respect to the semi-norm. So by Theorem 2.10 (see Remark 2.12) we have that ϕ is a quasismymetry.

This implies that g and f have quasiconformal extensions to $\overline{\mathbb{C}}$. This is a consequence of the proof of the conformal welding theorem, but not the statement, so we supply the argument. We will use the following fact: given a quasicircle $\gamma \subset \overline{\mathbb{C}}$, a continuous map Φ of $\overline{\mathbb{C}}$ which is quasiconformal on the complements of γ is quasiconformal on $\overline{\mathbb{C}}$.

Let $w_\mu : \mathbb{D}^- \rightarrow \mathbb{D}^-$ be a quasiconformal extension of ϕ , which exists by the Beurling-Ahlfors extension theorem. Let μ be the Beltrami differential of w_μ . Let $w^\mu : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ be a solution to the Beltrami equation with Beltrami differential μ on \mathbb{D}^- and 0 on \mathbb{D}^+ . Define

the map $\Phi : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ to be the continuous extension of

$$\Phi(z) = \begin{cases} g \circ w_\mu \circ (w^\mu)^{-1}(z) & z \in w^\mu(\mathbb{D}^-) \\ f \circ (w^\mu)^{-1}(z) & z \in w^\mu(\mathbb{D}^+). \end{cases}$$

A continuous extension exists, since $(w^\mu)^{-1}$ is continuous on the omitted set $\gamma = w^\mu(\mathbb{S}^1)$ (which is a quasicircle), and $g \circ w_\mu = f$ on \mathbb{S}^1 by definition of w_μ . Thus Φ is quasiconformal, and since γ is a quasicircle, $\Gamma = \Phi(\gamma)$ is also a quasicircle. This completes the proof. \square

Remark 2.15. Note that we do not claim that the quasisymmetry ϕ in the proof takes sets of harmonic measure zero to sets of harmonic measure zero. We are grateful to the referee for drawing our attention to this subtlety.

The following immediate consequence deserves to be singled out, since it will allow us to consistently define the set of boundary values of harmonic functions of finite Dirichlet energy.

Corollary 2.16. *Let $\Gamma \subset \overline{\mathbb{C}}$ be a quasicircle. Then $\mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$.*

That is, $\mathcal{D}_{\text{harm}}(\Omega^+)$ and $\mathcal{D}_{\text{harm}}(\Omega^-)$ have the same boundary values.

Theorem 2.14 also holds for the pointed norms.

Theorem 2.17. *Let Γ be a closed Jordan curve bounding Ω^\pm . For any fixed $p^\pm \in \Omega^\pm$, the maps $\mathfrak{R}^{+-}(\Gamma) : \mathcal{D}_{\text{harm},p^+}(\Omega^+) \rightarrow \mathcal{D}_{\text{harm},p^-}(\Omega^-)$ and $\mathfrak{R}^{-+}(\Gamma) : \mathcal{D}_{\text{harm},p^-}(\Omega^-) \rightarrow \mathcal{D}_{\text{harm},p^+}(\Omega^+)$ are bounded if and only if Γ is a quasicircle.*

Proof. Choose $f : \mathbb{D}^+ \rightarrow \Omega^+$ and $g : \mathbb{D}^- \rightarrow \Omega^-$ to be conformal maps such that $f(0) = p^+$ and $g(\infty) = p^-$. We define extension and restriction operators r_\pm and e_\pm as in the proof of Theorem 2.14 except that now we observe that they are isometries between $\mathcal{D}_{\text{harm}}(\mathbb{D}^\pm)$ and $\mathcal{H}(\mathbb{S}^1)$ with respect to the $\|\cdot\|_{\mathcal{H}_0(\mathbb{S}^1)}$ and $\|\cdot\|_{\mathcal{D}_{\text{harm},q^\pm}(\mathbb{D}^\pm)}$ norms, where $q^+ = 0$ and $q^- = \infty$. The proof proceeds as in Theorem 2.14.

Conversely for the welding map $\phi = g^{-1} \circ f$ the proof of Theorem 2.14 shows that that $C_{\phi^{-1}} = r_- C_g \mathfrak{R}^{+-}(\Gamma) C_{f^{-1}e_+}$, so the boundedness of $\mathfrak{R}^{+-}(\Gamma)$ will yield the boundedness of $C_{\phi^{-1}}$ on $\mathcal{H}(\mathbb{S}^1)$, which by Theorem 2.13 implies that ϕ^{-1} (and therefore also ϕ) is a quasisymmetry. Repeating the conformal welding argument in the proof of Theorem 2.14 we have that Γ is a quasicircle. Following a similar argument, the boundedness of $\mathfrak{R}^{-+}(\Gamma)$ also implies that Γ is a quasicircle. \square

Theorem 2.14 has the following important consequence. For any quasicircle Γ bounding domains Ω^\pm , there is a constant $C > 0$ such that for any $h \in \mathcal{H}(\Gamma)$ the extensions $H_\pm \in \mathcal{D}_{\text{harm}}(\Omega^\pm)$ satisfy

$$(2.5) \quad \frac{1}{C} \|H_-\|_{\mathcal{D}_{\text{harm}}(\Omega^-)} \leq \|H_+\|_{\mathcal{D}_{\text{harm}}(\Omega^+)} \leq C \|H_-\|_{\mathcal{D}_{\text{harm}}(\Omega^-)}.$$

Thus we may make the following definition:

Definition 2.18. Let Γ be a quasicircle in $\overline{\mathbb{C}}$. Define the ‘‘Douglas-Osborn space’’ by $\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma) = \mathcal{H}_-(\Gamma)$.

Remark 2.19. Thus, for a quasicircle, any statement regarding boundedness with respect to either $\mathcal{H}_+(\Gamma)$ or $\mathcal{H}_-(\Gamma)$ is automatically true for both semi-norms.

It is a remarkable property of quasicircles that the spaces are the same and the norms are equivalent.

We note that neither Douglas nor Osborn made special mention of quasicircles. However, we chose the name because of their pioneering work on expressing the Dirichlet norm in terms of boundary values.

We can also define a pointed norm on the Douglas-Osborn space as follows.

Definition 2.20. Let Γ be a quasicircle in $\overline{\mathbb{C}}$. For $p \notin \Gamma$, define the pointed Douglas-Osborn norm as follows. Let Ω be the component of the complement of Γ containing p . For $h \in \mathcal{H}(\Gamma)$ let H be the unique harmonic extension in $\mathcal{D}_{\text{harm}}(\Omega)$. Define

$$\|h\|_{\mathcal{H}_p(\Gamma)} = \|H\|_{\mathcal{D}_{\text{harm},p}(\Omega)}.$$

Of course, this choice is not canonical and breaks the symmetry between inside and outside domain. On the other hand, this defines an actual norm as opposed to a semi-norm.

We also immediately have that the Douglas-Osborn semi-norm is conformally invariant in the following sense. If Γ_1 and Γ_2 are quasicircles bounding domains Ω_1 and Ω_2 respectively, then a conformal map $f : \Omega_1 \rightarrow \Omega_2$ has a unique homeomorphic extension to Γ_1 and Γ_2 . We therefore have a well-defined composition map

$$\begin{aligned} C_f : \mathcal{H}(\Gamma_2) &\rightarrow \mathcal{H}(\Gamma_1) \\ h &\mapsto h \circ f. \end{aligned}$$

Then we have the following immediate consequence of our definitions.

Theorem 2.21. *Let Γ_1 and Γ_2 be Jordan curves in $\overline{\mathbb{C}}$, and let Ω_1^\pm and Ω_2^\pm be components of their complements respectively. If $f^\pm : \Omega_1^\pm \rightarrow \Omega_2^\pm$ is a conformal map, then $C_{f^\pm} : \mathcal{H}(\Gamma_2) \rightarrow \mathcal{H}(\Gamma_1)$ is an isometry with respect to $\|\cdot\|_{\mathcal{H}_\pm(\Gamma_i)}$.*

If $p_1^\pm \in \Omega_1^\pm$ and $p_2^\pm = f(p_1^\pm) \in \Omega_2^\pm$, then C_{f^\pm} is an isometry with respect to $\|\cdot\|_{\mathcal{H}_{p_i^\pm}(\Gamma_i)}$.

3. JUMP DECOMPOSITION AND WELL-POSEDNESS OF A RIEMANN-HILBERT PROBLEM

3.1. Cauchy-type operators on quasidisks. In this section we define a limiting Cauchy integral, and show that it is a bounded map from $\mathcal{H}(\Gamma)$ to the holomorphic Dirichlet spaces of the complement. By holomorphic Dirichlet spaces we mean, for a Jordan domain $\Omega \subset \overline{\mathbb{C}}$,

$$\mathcal{D}(\Omega) = \{h \in \mathcal{D}_{\text{harm}}(\Omega) : h \text{ is holomorphic}\}.$$

The semi-norm and norm on the harmonic Dirichlet space restrict to $\mathcal{D}(\Omega)$; we denote these by

$$\|h\|_{\mathcal{D}(\Omega)}^2 = \iint_{\Omega} |h'|^2 dA$$

and

$$\|h\|_{\mathcal{D}_p(\Omega)}^2 = |h(p)|^2 + \iint_{\Omega} |h'|^2 dA.$$

For convenience in our Cauchy-type integral operators, throughout the rest of the paper we will assume that Γ does not contain ∞ . Recall that according to our conventions, Ω^- will be the unbounded component of the complement of Γ . For the unbounded domain, we will define

$$\mathcal{D}_*(\Omega^-) = \{h \in \mathcal{D}(\Omega^-) : h(\infty) = 0\}.$$

Of course the restriction of $\|\cdot\|_{\mathcal{D}(\Omega^-)}$ is a norm on $\mathcal{D}_*(\Omega^-)$.

We define an operator which will play the role of the Cauchy integral in the Riemann-Hilbert boundary value problem. We will see in Theorem 3.5 ahead that it equals a Cauchy integral in a certain sense. It is important to note that in what follows, we will make statements and claims that are valid for general Jordan curves. We shall therefore emphasize that fact by including that assumption in all the statements below, until otherwise stated explicitly.

Definition 3.1. Let Γ be a Jordan curve not containing ∞ , bounding Ω^\pm . We define the jump operator on $\mathcal{H}_+(\Gamma)$ as follows. For $h \in \mathcal{H}_+(\Gamma)$ let $h_{\Omega^+} \in \mathcal{D}_{\text{harm}}(\Omega^+)$ be its unique extension. Then

$$(3.1) \quad J(\Gamma)h(z) := h_{\Omega^+}(z)\chi_{\Omega^+}(z) + \frac{1}{\pi} \iint_{\Omega^+} \frac{\bar{\partial}h_{\Omega^+}(\zeta)}{\zeta - z} dA(\zeta), \quad z \in \mathbb{C} \setminus \Gamma$$

where χ_{Ω^+} denotes the characteristic function of the closure of Ω^+ .

Denoting by $|\Omega^+|$ the area of Ω^+ , which is finite because Ω^+ is bounded, we have the estimate

$$(3.2) \quad \iint_{\Omega^+} |\bar{\partial}h_{\Omega^+}(\zeta)| dA(\zeta) \leq |\Omega^+|^{\frac{1}{2}} \left(\iint_{\Omega^+} |\bar{\partial}h_{\Omega^+}(\zeta)|^2 dA(\zeta) \right)^{\frac{1}{2}} \leq |\Omega^+|^{\frac{1}{2}} \|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}$$

so $\bar{\partial}h_{\Omega^+} \in L^1(\Omega^+)$. It is also easily seen that the formula implies that $\lim_{z \rightarrow \infty} J(\Gamma)h(z) = 0$, and it is understood that $J(\Gamma)h(z)$ extends to a function on $\Omega^+ \sqcup \Omega^-$.

To prove that $J(\Gamma)$ is bounded, we also need estimates for a certain integral operator. Initially, for $\varphi \in C_0^\infty(\Omega^+)$ (C^∞ with compact support in Ω^+) we define the operator T_{Ω^+} via

$$(3.3) \quad T_{\Omega^+}\varphi(z) = \frac{1}{\pi} \iint_{\Omega^+} \frac{\varphi(\zeta)}{\zeta - z} dA(\zeta).$$

One has that

$$(3.4) \quad \|T_{\Omega^+}\varphi\|_{L^2(\Omega^+)} \leq C\|\varphi\|_{L^2(\Omega^+)}$$

and

$$(3.5) \quad \|\partial(T_{\Omega^+}\varphi)\|_{L^2(\Omega^\pm)} + \|\bar{\partial}(T_{\Omega^+}\varphi)\|_{L^2(\Omega^\pm)} \leq C\|\varphi\|_{L^2(\Omega^+)}.$$

The estimate (3.4) was proven in [15, Lemma 3.1]. Estimate (3.5) is a direct consequence of the facts that $\partial(T_{\Omega^+}\varphi)(z) = \text{P.V.} \iint_{\Omega^+} \frac{\varphi(\zeta)}{(\zeta - z)^2} dA(\zeta)$, which is the Beurling transform of $\varphi\chi_{\Omega^+}$, and $\bar{\partial}(T_{\Omega^+}\varphi)(z) = -\varphi(z)\chi_{\Omega^+}(z)$. Both derivatives here are in the sense of distributions. Now since the Beurling transform $B\varphi(z) := \text{P.V.} \iint_{\mathbb{C}} \frac{\varphi(\zeta)}{(\zeta - z)^2} dA(\zeta)$ is a Calderón-Zygmund singular integral operator, it is well-known that B is bounded on $L^p(\mathbb{C})$ for $1 < p < \infty$. See e.g. [8] for the proof of these facts.

Now we can state and prove the following key result.

Theorem 3.2. *Let Γ be a Jordan curve not containing ∞ , bounding Ω^\pm . Then $J(\Gamma)h$ is holomorphic in $\Omega^+ \sqcup \Omega^-$ for all $h \in \mathcal{H}_+(\Gamma)$. Furthermore the operators*

$$\begin{aligned} P(\Omega^+) : \mathcal{H}_+(\Gamma) &\rightarrow \mathcal{D}(\Omega^+) \\ h &\mapsto J(\Gamma)h|_{\Omega^+} \end{aligned}$$

and

$$\begin{aligned} P(\Omega^-) : \mathcal{H}_+(\Gamma) &\rightarrow \mathcal{D}(\Omega^-) \\ h &\mapsto -J(\Gamma)h|_{\Omega^-} \end{aligned}$$

are bounded with respect to the semi-norms $\|\cdot\|_{\mathcal{H}_+(\Gamma)}$, $\|\cdot\|_{\mathcal{D}(\Omega^+)}$ and $\|\cdot\|_{\mathcal{D}(\Omega^-)}$.

Proof. For $h \in \mathcal{H}_+(\Gamma)$, set

$$h_{\pm} = \pm(J(\Gamma)h)|_{\Omega^{\pm}}.$$

We first show that h_{\pm} are holomorphic in Ω^{\pm} . Since $J(\Gamma)h = h_{\Omega^+} + T_{\Omega^+}\bar{\partial}h_{\Omega^+}$ one has that $\bar{\partial}J(\Gamma)h(z) = \bar{\partial}h_{\Omega^+} - \bar{\partial}h_{\Omega^+} = 0$. On Ω^- one has

$$J(\Gamma)h(z) = \frac{1}{\pi} \iint_{\Omega^+} \frac{\bar{\partial}h_{\Omega^+}(\zeta)}{\zeta - z} dA(\zeta),$$

which is obviously holomorphic on $\Omega^- \setminus \{\infty\}$. From the expression (3.1) we also see that $J(\Gamma)h \rightarrow 0$ as $z \rightarrow \infty$, so in fact $J(\Gamma)h$ is holomorphic in a neighbourhood of ∞ .

Next we show that $h \mapsto h_{\pm}$ are bounded. The estimate (3.5) for Ω^{\pm} yields that

$$(3.6) \quad \|h_{+}\|_{\mathcal{D}(\Omega^+)} = \|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}(\Omega^+)} \leq \|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)} + C\|\bar{\partial}h_{\Omega^+}\|_{L^2(\Omega^+)} \leq C\|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}$$

and

$$(3.7) \quad \|h_{-}\|_{\mathcal{D}(\Omega^-)} = \|(J(\Gamma)h)|_{\Omega^-}\|_{\mathcal{D}(\Omega^-)} \leq C\|\bar{\partial}h_{\Omega^+}\|_{L^2(\Omega^+)} \leq C\|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}.$$

which proves the claim. \square

Corollary 3.3. *Let Γ be a Jordan curve not containing ∞ , bounding Ω^{\pm} . Let $p \in \Omega^+$. $P(\Omega^+)$ is bounded with respect to the norms $\|\cdot\|_{\mathcal{H}_p(\Gamma)}$ and $\|\cdot\|_{\mathcal{D}_p(\Omega^+)}$; and $P(\Omega^-)$ is bounded with respect to $\|\cdot\|_{\mathcal{H}_p(\Gamma)}$ and $\|\cdot\|_{\mathcal{D}(\Omega^-)}$.*

Proof. The claim for $P(\Omega^-)$ follows from Theorem 3.2 and the fact that $\|\cdot\|_{\mathcal{H}(\Gamma)} \leq \|\cdot\|_{\mathcal{H}_p(\Gamma)}$. Therefore we shall only prove the claim for $P(\Omega^+)$. For the pointed norm we have

$$\begin{aligned} \|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}_p(\Omega^+)}^2 &= |J(\Gamma)h(p)|^2 + \|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}(\Omega^+)}^2 \\ &= |h_{\Omega^+}(p) + T\bar{\partial}h_{\Omega^+}(p)|^2 + \|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}(\Omega^+)}^2, \end{aligned}$$

where the operator T is defined by (3.3).

Through (3.6) we already know that $\|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}(\Omega^+)} \leq C\|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}$.

Moreover $T\bar{\partial}h_{\Omega^+}(z)$ is a harmonic function in Ω^+ , if h_{Ω^+} is harmonic in Ω^+ . This is because $\bar{\partial}(T\bar{\partial}h_{\Omega^+}(z)) = -\bar{\partial}h_{\Omega^+}(z)$, for $z \in \Omega^+$.

Now since $p \in \Omega^+$ and $T\bar{\partial}h_{\Omega^+}$ is harmonic in Ω^+ , there is an $r > 0$ such that $\mathbb{D}(p, r) \subset \Omega^+$ and by the mean-value theorem for harmonic functions one has

$$(3.8) \quad |T\bar{\partial}h_{\Omega^+}(p)| \leq \frac{1}{\pi r^2} \iint_{\mathbb{D}(p, r)} |T\bar{\partial}h_{\Omega^+}(z)| dA(z).$$

Moreover Jensen's inequality yields

$$(3.9) \quad |T\bar{\partial}h_{\Omega^+}(p)|^2 \leq \frac{|\Omega^+|}{\pi^2 r^4} \iint_{\Omega^+} |T\bar{\partial}h_{\Omega^+}(z)|^2 dA(z).$$

Hence, using estimates (3.8) and (3.9), together with the L^2 -estimate (3.4), we have

$$(3.10) \quad |T\bar{\partial}h_{\Omega^+}(p)|^2 \leq C\|T\bar{\partial}h_{\Omega^+}\|_{L^2(\Omega^+)}^2 \leq C\|\bar{\partial}h_{\Omega^+}\|_{L^2(\Omega^+)}^2 \leq C\|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}^2.$$

Finally gathering all the estimates, we obtain

$$(3.11) \quad \|(J(\Gamma)h)|_{\Omega^+}\|_{\mathcal{D}_p(\Omega^+)}^2 \leq C(|h_{\Omega^+}(p)|^2 + \|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm}}(\Omega^+)}^2) = C\|h_{\Omega^+}\|_{\mathcal{D}_{\text{harm},p}(\Omega^+)}^2,$$

and Definition 2.20 ends the proof of this corollary. \square

Remark 3.4. In the notations $P(\Omega^\pm)$ and $J(\Gamma)$, P of course stands for projection and J stands for jump.

At this point we return to the realm of quasicircles. The following limiting integral expression is key to the results of the next section.

Theorem 3.5. *Let Γ be a quasicircle, not containing ∞ , bounding domains Ω^\pm . Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ and $g : \mathbb{D}^- \rightarrow \Omega^-$ be conformal maps. Let γ_r denote the curve $|w| = r$ traced counter-clockwise. Then for all $h \in \mathcal{H}(\Gamma)$ and all $z \in \mathbb{C} \setminus \Gamma$,*

$$(3.12) \quad \begin{aligned} J(\Gamma)h(z) &= \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta \\ &= \lim_{r \searrow 1} \frac{1}{2\pi i} \int_{g(\gamma_r)} \frac{h_{\Omega^-}(\zeta)}{\zeta - z} d\zeta. \end{aligned}$$

Proof. We prove the first integral formula, which is straightforward. For $z \in \Omega^-$, it follows directly from Stokes' theorem. If $z \in \Omega^+$ we apply the mean value theorem for harmonic functions to the first term of (3.1) and the Stokes theorem to its second term to obtain

$$\begin{aligned} J(\Gamma)h(z) &= \lim_{R \searrow 0} \frac{1}{2\pi i} \int_{|\zeta-z|=R} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta + \lim_{r \nearrow 1} \lim_{R \searrow 0} \frac{1}{\pi} \iint_{f(\{|w|<r\}) \setminus \{|\zeta-z|\leq R\}} \frac{\bar{\partial}h_{\Omega^+}(\zeta)}{\zeta - z} dA(\zeta) \\ &= \lim_{R \searrow 0} \frac{1}{2\pi i} \int_{|\zeta-z|=R} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta + \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(|w|=r)} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta \\ &\quad - \lim_{R \searrow 0} \frac{1}{2\pi i} \int_{|\zeta-z|=R} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta = \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

which proves the first integral expression.

The second integral expression requires more work, and proceeds as follows. We first show that the two limiting integrals are equal for those elements of the Dirichlet space which are smooth on the closure. By applying boundedness of the reflection and density of this subset, the result follows. To prove the result for smooth harmonic functions we require two preparatory facts. Let U be an open set containing the quasicircle Γ . Let $q(z)$ be a smooth function on U . For $\epsilon > 0$, let V_ϵ denote the region bounded by the analytic curves $f(\gamma_{1-\epsilon})$ and $g(\gamma_{1+\epsilon})$. Since Γ is a quasicircle, it has Lebesgue measure zero (since quasiconformal

maps take Lebesgue null-sets to Lebesgue null-sets, see e.g. [8, I.3.5]). Thus by Stokes' theorem

$$(3.13) \quad \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \left(\int_{f(\gamma_{1-\epsilon})} \frac{q(\zeta)}{\zeta - z} d\zeta - \int_{g(\gamma_{1+\epsilon})} \frac{q(\zeta)}{\zeta - z} d\zeta \right) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{V_\epsilon} \frac{\bar{\partial} q(\zeta)}{\zeta - z} dA_\zeta = 0.$$

Next, observe that if $Q(z)$ is continuous on $U \cap (\Omega^+)^c$ (where \cdot^c denotes complement) and zero on Γ then

$$(3.14) \quad \lim_{\epsilon \rightarrow 0} \int_{g(\gamma_{1+\epsilon})} \frac{Q(\zeta)}{\zeta - z} d\zeta = 0, \quad z \in \overline{\mathbb{C}} \setminus \Gamma$$

Now let $W = U \cup \Omega^+$. The claim is now easily seen to hold for harmonic functions on Ω^+ which are smooth on W . Let p be such a function, and let h denote its boundary values. Thus $h_{\Omega^+} = p|_{\Omega^+}$. Furthermore we have that $h_{\Omega^-} = \mathfrak{R}^{+-}(p|_{\Omega^+})$ has a continuous extension to $\text{cl}\Omega^-$ with respect to the spherical topology. To see this, first we apply a translation to arrange that Ω^- does not contain 0 in its closure. We observe then that $h_{\Omega^-}(1/z)$ is the solution to the Dirichlet problem on the bounded domain $1/\Omega^-$ with continuous boundary data on $1/\Gamma$, so it is continuous on the closure of $1/\Omega^-$; the claim now follows from the facts that the Euclidean and spherical topologies are the same on bounded domains, and that $1/z$ and translations are continuous in the spherical topology. In particular, h_{Ω^-} is continuous on $V_{\epsilon_0} \cap (\Omega^+)^c \subset W$ for some $\epsilon_0 > 0$, (where containment in W is obtained by choosing ϵ_0 sufficiently small). Thus since $p - h_{\Omega^-}$ is continuous on $V_{\epsilon_0} \cap (\Omega^+)^c$ and zero on Γ , by (3.14)

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{g(\gamma_{1+\epsilon})} \frac{h_{\Omega^-}(\zeta)}{\zeta - z} d\zeta &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{g(\gamma_{1+\epsilon})} \frac{p(\zeta)}{\zeta - z} d\zeta \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{f(\gamma_{1-\epsilon})} \frac{p(\zeta)}{\zeta - z} d\zeta \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi i} \int_{f(\gamma_{1-\epsilon})} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

where the second-to-last equality follows from (3.13).

Finally, we observe that harmonic functions which are smooth on some W as above are dense in $\mathcal{H}(\Gamma)$, since for example the set of holomorphic and anti-holomorphic polynomials is dense in $\mathcal{D}_{\text{harm}}(\Omega)$. To prove this, one can for example apply the density of polynomials in the Bergman space of so-called Carathéodory domains [9, v.3, Section 15] which includes quasidisks. By the fact that differentiation is an isometry between the holomorphic Dirichlet and Bergman space (up to constants), it follows that polynomials are dense in the holomorphic Dirichlet space. Similarly conjugate polynomials are dense in the complex conjugate of the holomorphic Dirichlet space. Since harmonic functions on a simply connected domain have a unique decomposition into holomorphic and antiholomorphic parts (up to constants), the set of $p(z) + \overline{q(z)}$ where p and q are polynomials is dense in the complex harmonic Dirichlet space. Since \mathfrak{R}^{+-} is a bounded operator, this completes the proof. \square

Note that the formulation of the previous theorem requires the fact that h has an extension to both inside and outside, and the proof requires that the reflections $\mathfrak{R}^{+-}(\Gamma)$ and $\mathfrak{R}^{-+}(\Gamma)$ are bounded. Only quasicircles have both properties.

Although the following corollary is now obvious, it is worth writing out explicitly. Using notation as in Theorem 3.5, for a quasicircle Γ not containing ∞ we can define two Cauchy

integral-type projections on $\mathcal{H}(\Gamma)$ in two distinct reasonable ways using limiting integrals:

$$\Pi_+(\Omega^\pm) : \mathcal{H}(\Gamma) \rightarrow \mathcal{D}(\Omega^\pm)$$

$$h \mapsto \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta$$

and

$$\Pi_-(\Omega^\pm) : \mathcal{H}(\Gamma) \rightarrow \mathcal{D}(\Omega^\pm)$$

$$h \mapsto -\lim_{r \searrow 1} \frac{1}{2\pi i} \int_{g(\gamma_r)} \frac{h_{\Omega^-}(\zeta)}{\zeta - z} d\zeta.$$

Corollary 3.6. *Let Γ be a quasicircle not containing ∞ , bounding domains Ω^\pm . Then $\Pi_+(\Omega^\pm) = \Pi_-(\Omega^\pm)$. Furthermore, $\Pi_+(\Omega^\pm)$ and $\Pi_-(\Omega^\pm)$ are each bounded with respect to both norms $\mathcal{H}_+(\Gamma)$ and $\mathcal{H}_-(\Gamma)$ on the domain.*

Proof. This follows from Theorem 3.2, (2.5) and Theorem 3.5. \square

Remark 3.7. Of course, the integral operators (3.12) define bounded operators directly from $\mathcal{D}_{\text{harm}}(\Omega^\pm)$ with respect to either the semi-norms or norms.

Remark 3.8. In defining the operator $J(\Gamma)$, we “broke the symmetry” between the inside and outside by choosing to take the integral over Ω^+ . Theorem 3.5 can be used to show that for quasicircles, $J(\Gamma)$ can be defined using the integral over Ω^-

$$J(\Gamma)h(z) := h_{\Omega^-}(z)\chi_{\Omega^-}(z) + \frac{1}{\pi} \iint_{\Omega^-} \frac{\bar{\partial} h_{\Omega^-}(\zeta)}{\zeta - z} dA(\zeta), \quad z \in \mathbb{C} \setminus \Gamma$$

with no change to the outcome (we leave the proof to the interested reader). This is obvious for highly regular curves by Stokes’ theorem, but for quasicircles it is a surprisingly subtle point. The proof of Theorem 3.5 suggests that it is closely linked to the density of polynomials in the Dirichlet space of a quasicircle.

Remark 3.9. It is also possible to prove a version of Corollary 3.3 where the pointed norm is taken on the outside. However, to do so one must alter the normalization of the Cauchy kernel so that $P(\Omega^+)h(p) = 0$ for some point $p \in \Omega^+$, rather than $P(\Omega^-)h(\infty) = 0$ as we have here.

3.2. The jump decomposition on quasicircles. In this section we prove that the jump decomposition on quasicircles holds. To do this, we will first show that $\mathcal{H}(\Gamma)$ is naturally isomorphic to $\mathcal{D}(\Omega^+) \oplus \mathcal{D}_*(\Omega^-)$. The actual jump formula follows easily.

We require a theorem.

Theorem 3.10. *Let Γ be a quasicircle not containing ∞ , bounding domains Ω^\pm . For any $h \in \mathcal{H}(\Gamma)$ such that $h_{\Omega^-} \in \mathcal{D}_*(\Omega^-)$, we have that*

$$J(\Gamma)h(z) = \begin{cases} 0 & z \in \Omega^+ \\ -h_{\Omega^-}(z) & z \in \Omega^- \end{cases}.$$

On the other hand, for any $h \in \mathcal{H}(\Gamma)$ such that $h_{\Omega^+} \in \mathcal{D}(\Omega^+)$

$$J(\Gamma)h(z) = \begin{cases} h_{\Omega^+}(z) & z \in \Omega^+ \\ 0 & z \in \Omega^- \end{cases}.$$

Proof. We apply the classical Plemelj-Sokhotski jump formula on regular curves and use the limiting integral formula. For a holomorphic function h on a domain Ω bounded by a smooth Jordan curve γ , if h extends continuously to γ then the classical formula yields

$$\frac{1}{2\pi i} \int_{\gamma} \frac{h(\zeta)}{\zeta - z} d\zeta = \begin{cases} h(z) & z \in \Omega \\ 0 & z \in \mathbb{C} \setminus \text{cl}\Omega \end{cases}$$

and similarly for h holomorphic in the complementary region and vanishing at ∞ .

Now assume that h is the boundary values of some h_{Ω^+} in the sense of Osborn. By Theorem 3.5, fixing z we have

$$J(\Gamma)h(z) = \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h_{\Omega^+}(\zeta)}{\zeta - z} d\zeta = \begin{cases} h_{\Omega^+}(z) & z \in \Omega^+ \\ 0 & z \in \Omega^- \end{cases}$$

since for any $z \notin \Gamma$, the curve $f(\gamma_r)$ eventually lies either always inside or always outside $f(\gamma_r)$ on some interval $[R, 1)$ for $R < 1$, and thus the integral is independent of r ; for any fixed $r \in [R, 1)$ we can apply the classical formula. The other claim is proven similarly using the integral formula involving the conformal map $g : \mathbb{D}^- \rightarrow \Omega^-$. \square

We will need some identities to prove the jump decomposition. Let Γ be a quasicircle not containing ∞ , bounding domains Ω^\pm . We define the mapping

$$\begin{aligned} \mathbf{I}_f : \mathcal{D}_*(\mathbb{D}^-) &\rightarrow \mathcal{D}_*(\Omega^-) \\ h &\mapsto P(\Omega^-)C_{f^{-1}}h. \end{aligned}$$

There is a restriction to the boundary which we have suppressed for simplicity of notation. We know that this is a bounded map into $\mathcal{D}_*(\Omega^-)$ by Theorems 3.2 and 2.21.

Theorem 3.11. *Let Γ be a Jordan curve not containing ∞ , with bounded and unbounded components Ω^\pm respectively. Let $f : \mathbb{D}^+ \rightarrow \Omega^+$ be a conformal map. For any polynomial $h \in \mathbb{C}[1/z]$ with zero constant term,*

$$\mathbf{I}_f h(z) = - \lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h \circ f^{-1}(\zeta)}{\zeta - z} d\zeta.$$

Proof. By linearity it is enough to verify this for monomials. Setting $h(z) = z^{-n}$ for $n > 0$, it is easily verified that

$$(h \circ f^{-1})_{\Omega^+}(z) = \overline{h \circ (1/f^{-1}(z))}$$

since the boundary values agree on \mathbb{S}^1 after composition by f . Thus

$$\begin{aligned}
(\mathbf{I}_f h)(z) &= -\lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{\overline{f^{-1}(\zeta)}^n}{\zeta - z} d\zeta \\
&= -\lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\gamma_r} \frac{\bar{\xi}^n}{f(\xi) - z} f'(\xi) d\xi \\
&= -\lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\gamma_r} \frac{r^{2n} \xi^{-n}}{f(\xi) - z} f'(\xi) d\xi \\
&= -\lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{\gamma_r} \frac{\xi^{-n}}{f(\xi) - z} f'(\xi) d\xi \\
&= -\lim_{r \nearrow 1} \frac{1}{2\pi i} \int_{f(\gamma_r)} \frac{h \circ f^{-1}(\zeta)}{\zeta - z} d\zeta
\end{aligned}$$

where we have used the fact that the integral $\int_{\gamma_r} \xi^{-n} \cdot f'(\xi)/(f(\xi) - z) d\xi$ is independent of r . \square

The map \mathbf{I}_f is closely related to Faber polynomials, whose definition we now recall.

Definition 3.12. Let $f : \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic map which is one-to-one on a neighbourhood of 0. Let $p = f(0)$. The n th Faber polynomial of f is

$$(3.15) \quad \Phi_n(f)(z) = \frac{c_{-n}}{(z-p)^n} + \cdots + \frac{c_{-1}}{(z-p)}$$

where the c_n are defined by

$$(f^{-1}(z))^{-n}(z) = \sum_{k=-n}^{\infty} c_k(z-p)^k.$$

Remark 3.13. Usually, it is assumed that $p = 0$ in the literature. Allowing $p \neq 0$ costs nothing and we will require it for use in future papers. It is also common to include the constant term in the expansion (3.15) in the Faber polynomials.

By Theorem 3.11, we have that

$$(3.16) \quad \mathbf{I}_f(z^{-n}) = \Phi_n.$$

The Faber polynomials satisfy the following identity.

$$(3.17) \quad \Phi_n(f) \circ f(z) = z^{-n} + \sum_{k=0}^{\infty} \beta_k^n z^k$$

for some coefficients β_k^n referred to as the Grunsky coefficients [4, 13] (usually in the case that $p = 0$). To prove (3.17), observe that $\Phi_n(f)(\zeta) = f^{-1}(\zeta)^{-n} - H(\zeta)$ where $H = P(\Omega^+)((f^{-1})^{-n})$ is holomorphic on Ω^+ . Thus $\Phi_n(f)(f(z)) = z^{-n} - H \circ f(z)$ where $H \circ f$ is holomorphic on \mathbb{D}^+ . Since f is holomorphic on $|z| < 1$ and $\Phi_n(f)$ is holomorphic on $0 < |z| < \infty$, the radius of convergence of the power series on the right hand side is greater than or equal to one. This leads to the following Lemma.

Lemma 3.14. *Let Γ be a quasicircle not containing ∞ . The map $P(\mathbb{D}^-)C_f$ is a bounded left inverse of \mathbf{I}_f . In particular, \mathbf{I}_f is injective.*

Proof. By Theorem 3.11 and equations (3.16) and (3.17) this holds for polynomials in $\mathbb{C}[1/z]$ with zero constant term. Since these are dense in $\mathcal{D}(\mathbb{D}^-)$ and $P(\mathbb{D}^-)C_f$ and \mathbf{I}_f are bounded, this completes the proof. \square

We may finally prove the main result of the paper.

Theorem 3.15. *For a quasicircle Γ not containing ∞ , bounding domains Ω^\pm , and $p \in \Omega^+$ define*

$$K : \mathcal{H}(\Gamma) \rightarrow \mathcal{D}(\Omega^+) \oplus \mathcal{D}_*(\Omega^-)$$

$$h \mapsto (P(\Omega^+)h, P(\Omega^-)h).$$

Then K is a bounded isomorphism with respect to the norms $\|\cdot\|_{\mathcal{H}_p(\Gamma)}$ and $\|\cdot\|_{\mathcal{D}_p(\Omega^+) \oplus \mathcal{D}(\Omega^-)}$.

Proof. The fact that K is bounded follows immediately from either Theorem 3.2 or Corollary 3.3, depending on the desired choice of norms. To see that it is surjective, given $(h_+, h_-) \in \mathcal{D}_p(\Omega^+) \oplus \mathcal{D}_*(\Omega^-)$, let \tilde{h}_+ and \tilde{h}_- denote their boundary values in $\mathcal{H}(\Gamma)$ and set $h = \tilde{h}_+ + \tilde{h}_-$. By Theorem 3.10, we have that $P(\Omega^\pm)h = h_\pm$ which proves the claim.

To show that K is injective, choose a conformal map $f : \mathbb{D}^+ \rightarrow \Omega^+$ such that $f(0) = p$. Observe that $C_{f^{-1}}$ is an isometry from $\mathcal{D}_0(\mathbb{D}^+)$ to $\mathcal{D}_p(\Omega^+)$ by Theorem 2.21. Hence it is an isometry from $\mathcal{H}_0(\mathbb{S}^1)$ to $\mathcal{H}_p(\Gamma)$. Thus any $h \in \mathcal{H}_p(\Gamma)$ can be written $C_{f^{-1}}(H_+ + H_-)$ for $H_+ \in \mathcal{D}_0(\mathbb{D}^+)$ and $H_- \in \mathcal{D}_*(\mathbb{D}^-)$. Now

$$(3.18) \quad Kh = (P(\Omega^+)C_{f^{-1}}H_+ + P(\Omega^+)C_{f^{-1}}H_-, P(\Omega^-)C_{f^{-1}}H_-).$$

If $Kh = 0$ then $\mathbf{I}_f H_- = P(\Omega^-)C_{f^{-1}}H_- = 0$ so by Lemma 3.14 $H_- = 0$. But then

$$0 = P(\Omega^+)C_{f^{-1}}H_+ + P(\Omega^+)C_{f^{-1}}H_- = P(\Omega^+)C_{f^{-1}}H_+.$$

Since $C_{f^{-1}}H_+ \in \mathcal{D}_p(\Omega^+)$ we have that $C_{f^{-1}}H_+ = P(\Omega^+)C_{f^{-1}}H_+ = 0$ but $C_{f^{-1}}$ is an isometry so $H_+ = 0$. Thus $h = 0$ so K is injective as claimed. \square

Remark 3.16. It is not hard to show that, conversely, K injective implies that \mathbf{I}_f is injective directly from the definition of K .

Remark 3.17. K is also bounded with respect to the semi-norms on $\mathcal{H}(\Gamma)$ and $\mathcal{D}(\Omega^\pm)$ by Theorem 3.2.

Finally, as a consequence of our previous results we establish the well-posedness of the following Riemann-Hilbert problem which was posed in less precise terms in the introduction of this paper.

Theorem 3.18. *Let Γ be a quasicircle not containing ∞ bounding Ω^\pm and let $h \in \mathcal{H}(\Gamma)$. Then there exist unique functions $u_+ \in \mathcal{D}(\Omega^+)$, $u_- \in \mathcal{D}_*(\Omega^-)$ such that the boundary values h_\pm of u_\pm are defined in the sense of Osborn and $h_+(w) + h_-(w) = h(w)$ except on a set of harmonic measure zero with respect to both Ω^+ and Ω^- . Moreover the solutions u_\pm depend continuously on h .*

Remark 3.19. The continuity of u_\pm is with respect to either the pointed norms $\mathcal{H}_p(\Gamma)$, $\mathcal{D}_p(\Omega^+)$, and $\mathcal{D}(\Omega^-)$, or to the unpointed norms $\mathcal{H}(\Gamma)$, $\mathcal{D}(\Omega^+)$, and $\mathcal{D}(\Omega^-)$.

Proof. We claim that $u_{\pm} = P(\Omega^{\pm})h$ is the unique solution to the problem. Indeed, $u_{+} \in \mathcal{D}(\Omega^{+})$ and $u_{-} \in \mathcal{D}_{*}(\Omega^{-})$ by Theorem 3.2, which also shows the continuous dependence of the solutions on the initial data h with respect to the semi-norms. The continuous dependence for the pointed norms follows from Corollary 3.3. By Theorem 3.10, $Kh_{+} = (u_{+}, 0)$, so $K(h - h_{+}) = (0, u_{-})$. Since $Kh_{-} = (0, u_{-})$ and K is injective we must have that $h - h_{+} = h_{-}$ which proves the Sokhotski-Plemelj jump relation. The uniqueness of the solution is a consequence of Theorem 3.15. \square

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